Connections between Theta-Graphs, Delaunay Triangulations, and Orthogonal Surfaces

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Abstract

Θ_k-graphs are geometric graphs that appear in the context of graph navigation. The shortest-path metric of these graphs is known to approximate the Euclidean complete graph up to a factor depending on the cone number k and the dimension of the space.

TD-Delaunay graphs, a.k.a. triangular-distance Delaunay triangulations introduced by Chew, have been shown to be plane 2-spanners of the 2D Euclidean complete graph, i.e., the distance in the TD-Delaunay graph between any two points is no more than twice the distance in the plane.

Orthogonal surfaces are geometric objects defined from independent sets of points of the Euclidean space. Orthogonal surfaces are well studied in combinatorics (orders, integer programming) and in algebra. From orthogonal surfaces, geometric graphs, called geodesic embeddings can be built.

In this paper, we introduce a specific subgraph of the Θ_6-graph defined in the 2D Euclidean space, namely the half-Θ_6-graph, composed of the even-cone edges of the Θ_6-graph. Our main contribution is to show that these graphs are exactly the TD-Delaunay graphs, and are strongly connected to the geodesic embeddings of orthogonal surfaces of coplanar points in the 3D Euclidean space.

Using these new bridges between these three fields, we establish:

- Every Θ_6-graph is the union of two spanning TD-Delaunay graphs. In particular, Θ_6-graphs are 2-spanners of the Euclidean graph, and the bound of 2 on the stretch factor is best possible. It was not known that Θ_6-graphs are t-spanners for some constant t, and Θ_7-graphs were only known to be t-spanners for $t \approx 7.562$.
- Every plane triangulation is TD-Delaunay realizable, i.e., every combinatorial plane graph for which all its interior faces are triangles is the TD-Delaunay graph of some point set in the plane. Such realizability property does not hold for classical Delaunay triangulations.

Keywords: Delaunay triangulation, theta-graph, orthogonal surface, spanner, realizability

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1 Introduction

A *geometric graph* is a weighted graph whose vertex set is a set of points of $\mathbb{R}^d$, and whose edge set consists in line segments joining two vertices. The weight of any edge is the Euclidean distance (L_2-norm) between its endpoints. The *Euclidean complete graph* is the complete geometric graph, in which all pairs of distinct vertices are connected by an edge.

Although geometric graphs are in theory specific weighted graphs, they naturally model many practical problems and in various fields of Computer Science, from Networking to Computational Geometry. Delaunay triangulations, Yao graphs, theta-graphs, β-skeleton graphs, Nearest-Neighborhood graphs, Gabriel graphs are just some of them [GO97]. A companion concept of the geometric graphs is the *graph spanner*. A *t-spanner* of a graph G is a spanning subgraph H such that for each pair u, v of vertices the distance in H between u and v is at most t times the distance in G between u and v. The value t is called *stretch factor* of the spanner.

Spanners have been independently introduced in Computational Geometry by Chew [Che86, Che89] for the complete Euclidean graph, and in the fields of Networking and Distributed Computing by Peleg and Ulman [PU87, PU89] for arbitrary graphs. Literature in connection with spanners is vast and applications are numerous. We refer to Peleg's book [Pel00], and Narasimhan and Smid's book [NS07] for comprehensive introduction to the topic.

1.1 Orthogonal surfaces

With a point set M of $\mathbb{R}^d$ it is possible to associate other geometric objects. Assuming that M consists only of pairwise incomparable[1] points, the *orthogonal surface* of M is the geometric boundary of the set of points of $\mathbb{R}^d$ greater to at least one point of M (see Fig. 2 for $d = 3$).

Orthogonal surfaces are rich mathematical objects with connections to various fields, including order dimension, integer programming, and monomial ideals. Schnyder woods and orthogonal surfaces of coplanar points of $\mathbb{R}^3$ have being established by Miller and Felsner et al. [Mil02, FZ08]. As a side effect, they gave an intuitive proof of the Brightwell-Trotter Theorem which is an extension to multigraphs of Schnyder's characterization of planar graphs in term of dimension of their incidence order [Sch89].

The *geodesic embedding* of a point set $S \subset \mathbb{R}^2$ is a geometric graph with vertex set S. To define its edges, one consider a specific embedding $\phi : S \to \mathbb{R}^3$ such that the points of $\phi(S)$ are coplanar (see Section 2). There is an edge between the points $p, q \in S$ if the *join* point $\phi(p) \vee \phi(q)$ belongs to the orthogonal surface of $\phi(S)$, the join point being defined as the point with maximum coordinates between $\phi(p)$ and $\phi(q)$ in each dimension.

1.2 Delaunay-graphs

In his seminal paper, Chew [Che86] has constructed plane spanners of the 2D Euclidean graph, namely planar subgraphs whose stretch factor is at most $\sqrt{10} \approx 3.162$. His construction is based on the L_1-Delaunay graph, i.e., the dual of the Voronoi diagram for the Manhattan distance (L_1-norm). He conjectured that L_2-Delaunay graphs, i.e., classical Delaunay triangulations, are t-spanners for some constant t. This conjecture has been proved by [DFS90], and the current best bounds on the stretch factor t of L_2-Delaunay graphs are $1.584 < t < 2.419$, proved respectively by [BDL$^+$09] and [KG92]. Determining the exact stretch factor of this important class of geometric graphs is a challenging and open question. We refer to the recent survey [BS09].

More generally, for any given convex set Γ in the plane[2], one can define the Γ-Delaunay graphs as the dual of the Voronoi diagram of a set of points with respect to the convex distance function defined by Γ. Bose et al. [BCCS08] have shown that Γ-Delaunay graphs are plane t-spanners for

[1] A point v is *greater* than u if, in for each dimension i, the ith v's coordinate is greater than the ith u's coordinate.

[2] To be more precise, Γ must be a compact and convex set with non-empty interior that contains its origin.

some stretch factor t depending on the shape of Γ.

A natural question, widely open, is to determine whether L_2-Delaunay graphs are the "best" plane spanners in term of stretch factor. Chew has proved in [Che86] that no plane t-spanner can exist if $t < \sqrt{2} \approx 1.414$, because of the square, a regular 4-gon. This lower bound has been slightly improved by [Mul04] with a regular 21-gon, showing that $t > 1.416$. On the upper bound side, Chew introduced in [Che89] the triangular distance-Delaunay graphs, *TD-Delaunay graphs* for short, whose convex distance function is defined from an equilateral triangle. He proved that TD-Delaunay graphs are plane 2-spanners. The stretch 2 is optimal with respect to the triangular distance because of some 3-gons.

1.3 Delaunay realizability

Searching for the "best" plane spanner should be done, a priori, in the set of all planar graphs. Indeed, there is no advantage to limit the search to any restricted subclass, excepted maybe to plane triangulations. By *plane triangulation* we mean a combinatorial plane graph in which all its interior faces are triangles. However, there are notorious plane triangulations that cannot be obtained from any L_2-Delaunay graphs, e.g., a K_4 for which a degree-3 vertex is added in each of its three interior faces [Dil90, LL97]. This leads to the question of *realizability* of plane triangulations by L_2-Delaunay graphs, and more generally by Γ-Delaunay graphs for a convex distance function Γ. More formally, a plane triangulation G is Γ-Delaunay *realizable* if there exists a point set S such that the Γ-Delaunay graph of S is isomorphic to G.

Every triangulation of any polygon, i.e., every maximal outerplane graph, can be realized by a L_2-Delaunay graph [Dil90]. Based on 3D hyperbolic geometry, Hodgson et al. [HRS92] gave a combinatorial characterization of the graphs that are L_2-Delaunay realizable, leading to a polynomial-time recognition algorithm by the use of integer programming. The algorithm has been simplified later in [HMS00, LP08]. Other connections between toughness, polyhedra of inscribable type, and L_2-Delaunay graphs have been developed in [DS96]. For arbitrary convex distance function D, the D-Delaunay realizability has not yet been studied.

1.4 Theta-graphs

Theta-graphs [Cla87, Kei88] and Yao graphs [Yao82] are very popular geometric graphs that appear in the context of navigating graphs. Adjacency is defined as follows: the space around each point p is decomposed into $k \geqslant 2$ regular cones, each with apex p, and a point $q \neq p$ of a given cone C is linked to p if, from p, q is the "nearest" point in C. When the points are in general positions, the out-degree is at most k and form a non-planar graph whenever $k > 6$.

Theta-graphs and Yao graphs differ in the way the nearest neighbor is defined. We focus on the 2D Euclidean space, so that each cone forms an angle of $\Theta_k = 2\pi/k$. For Yao graphs (Y_k-graphs for short), the nearest neighbor of p in the cone C is simply a point $q \neq p$ minimizing the L_2-distance between p and q. Whereas for theta-graphs (Θ_k-graphs for short), the nearest neighbor of p is the point whose orthogonal projection onto the bisector of C minimizes the L_2-distance.

Both graphs are known to be efficient spanners. The stretch factor of Θ_k-graphs and Y_k-graphs, proved respectively in [RS91] and in [Yao82], is at most $1/(1 - 2\sin(\pi/k))$ for every $k > 6$. Very little is known for $k \leqslant 6$. For instance, it is known that Y_4-graphs are connected [FLZ98], but it is only conjectured that Y_4-graphs are t-spanners for some constant t (see [DMP09, O'R09] for recent developments). For $k = 7$, and according to the current upper bound, we observe that the stretch of these graphs is larger than 7.562, and the stretch drops under 2 only from $k \geqslant 13$.

Our main result relies on a specific subgraph of the Θ_k-graph, namely the *half-Θ_k-graph*, taking only half the edges, those belonging to non consecutive cones in the counter-clockwise order (see Section 2 for more a formal definition). For even k, every Θ_k-graph is the union of two spanning half-Θ_k-graphs.

1.5 Our results

Our main contribution is an unexpected connection between TD-Delaunay graphs, orthogonal surfaces, and theta-graphs. We stress that these objects come from rather different domains and can lead to new results. We show that (see Section 3 for a more precise statement):

> *For every point set $S \subset \mathbb{R}^2$ in general position, the TD-Delaunay graph of S, the geodesic embedding of S, and the half-Θ_6-graph of S are equal.*

Half-Θ_6-graph turns out to be a key ingredient of our result. This unification result implies that each of these objects can directly inherit of all the known properties from the others. In particular, we exhibit two important corollaries:

1. *Every Θ_6-graph is the union of two spanning TD-Delaunay graphs.*

In particular, Θ_6-graphs are 2-spanners of the 2D Euclidean graph, because they contain a TD-Delaunay graph as spanning subgraph, which is a 2-spanner from [Che89]. Since the bound of 2 is optimal, we have therefore determined the stretch factor of Θ_6-graphs. Up to now, Θ_6-graphs were not known to be t-spanners for any constant t, and the best known bound on the stretch factor for Θ_7-graphs was larger than 7.562. Before this current paper, only Θ_k-graphs for $k \geqslant 13$ were known to be 2-spanners (see the previous best general upper bound [RS91]).

The other important consequence is:

2. *Every plane triangulation is TD-Delaunay realizable.*

We also show that the plane triangulation of K_4 is not L_1-Delaunay realizable, so that, to the best of our knowledge, the equilateral triangle is the only regular convex distance function D that is known to have the property that every plane triangulation is D-Delaunay realizable.

The paper is organized as follows. In Section 2 we precisely define all the objects we need, and in Section 3 we prove our main result. The corollaries are proved in Section 4.

2 Definitions

2.1 Half-Θ_6-graph

A *cone* is the region in the plane between two rays that emanate from the same point, its apex. For each cone C, let ℓ_C be the bisector ray of C, and for each point p, let $C^p = \{x + p : x \in C\}$.

Let us consider the rays obtained by a counter-clockwise rotation of the positive x-axis by angles of $2i\pi/k$ with integer i. Each pair of successive rays $2(i-1)\pi/k$ and $2i\pi/k$ defines a cone, denoted by A_i, whose apex is the origin. Let $\mathcal{A}_k = \{A_1, \ldots, A_k\}$.

The *directed* Θ_k*-graph* of a point set $S \subset \mathbb{R}^2$, denoted by $\overrightarrow{\Theta_k}(S)$, is defined as follows: (1) vertex set of $\overrightarrow{\Theta_k}(S)$ is S; and (2) (p, r) is an arc of $\overrightarrow{\Theta_k}(S)$ if and only if there is a cone $A_i \in \mathcal{A}_k$ such that $r \in A_i^p \setminus \{p\}$ whose orthogonal projection onto ℓ_C^p is the closest to p. We now introduce a new graph, called *half-*Θ_k*-graph*, defined as follows:

Definition 1 *The directed half-*Θ_k*-graph of a point set $S \subset \mathbb{R}^2$, denoted by $\frac{1}{2}\overrightarrow{\Theta_k}(S)$, is the digraph induced by all the arcs (p, r) of $\overrightarrow{\Theta_k}(S)$ such that $r \in A_i^p$ for some even number i.*

Whenever $k \equiv 2 \pmod 4$, we denote by C_i the cone A_{2i}, and by $\overline{C}_i$ the opposite cone of C_i, i.e., $\overline{C}_i = A_{2i+k/2 \bmod 6}$ (observe that $2i + k/2$ is odd). An arc (p, r) such that $r \in C_i^p$ is said to be colored i.

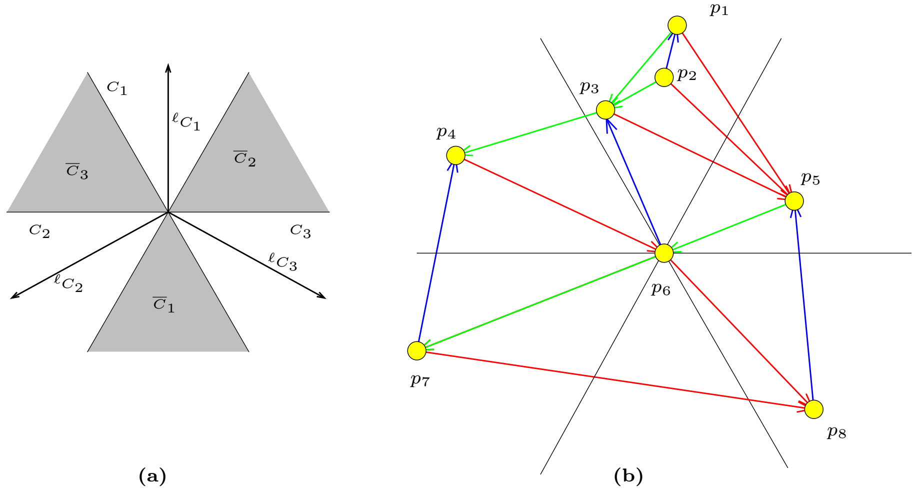

Figure 1: **(a)** Illustration of notations for half-Θ_6-graphs.**(b)** An example of a directed half-Θ_6-graph.

In this paper, we focus on the half-Θ_6-graph. So, in counter-clockwise order starting from the positive x-axis, the six cones of $\mathcal{A}_6$ are encountered in the order $\overline{C}_2, C_1, \overline{C}_3, C_2, \overline{C}_1, C_3$ (see Fig. 1(a)). Fig. 1(b) shows an example of a directed half-Θ_6-graph on 8 points.

The set of points S is said to be *degenerated* if there exist two points p and q in S such that both (p, q) and (q, p) are arcs of $\frac{1}{2}\overrightarrow{\Theta_k}(S)$. The set S is said to be *non-degenerated* otherwise.

2.2 Geodesic embeddings

Let $\mathcal{P}$ be a plane equipped with the standard basis $(\mathbf{e_x}, \mathbf{e_y})$, and let S be a finite set of points in the plane $\mathcal{P}$.

The following definitions are extracted from [Mil02]. Let $(\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3})$ be the standard basis of $\mathbb{R}^3$. The plane $\mathcal{P}$ is now embedded in $\mathcal{P}' \subset \mathbb{R}^3$ where $\mathcal{P}'$ is the plane containing the origin of $\mathbb{R}^3$ with basis $(\mathbf{e'_x}, \mathbf{e'_y})$ where $\mathbf{e'_x} = (0, -1/\sqrt{2}, 1/\sqrt{2})$ and $\mathbf{e'_y} = (\sqrt{2/3}, -1/\sqrt{6}, -1/\sqrt{6})$. Observe that $\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$ is a normal vector[3] of $\mathcal{P}'$. Any point $p = (p_x, p_y) \in \mathbb{R}^2$ is mapped to $p' \in \mathcal{P}'$ with $p' = p_x\mathbf{e'_x} + p_y\mathbf{e'_y}$.

Consider the dominance order on $\mathbb{R}^3$: $p \succcurlyeq q$ if and only if $p_i \geqslant q_i$ for each $i = 1, 2, 3$. Note that any two different points of $\mathcal{P}'$ are incomparable. The *filter* generated by a set of points S of $\mathcal{P}$ is the set

$$\langle S \rangle = \left\{ \alpha \in \mathbb{R}^3 : \alpha \succcurlyeq v \text{ for some } v \in S \right\} \ .$$

The boundary $\mathfrak{S}_S$ of $\langle S \rangle$ is the *coplanar orthogonal surface* generated by S. Notice that in [Mil02, FZ08], the authors also consider *orthogonal surfaces*, a more general case where elements of S are pairwise incomparable but not necessarily in the same plane of normal vector $\mathbf{e_1} + \mathbf{e_2} + \mathbf{e_3}$.

Fig. 2 shows an example of a coplanar orthogonal surface.

We denote by $p \vee q$ the point $(\max\{p_1, q_1\}, \max\{p_2, q_2\}, \max\{p_3, q_3\})$. If $p, q \in S$ and $p \vee q \in \mathfrak{S}_S$, then $\mathfrak{S}_S$ contains the union of the two line segments joining p and q to $p \vee q$. These lines are called *elbow geodesics* of $\mathfrak{S}_S$. The *orthogonal arc* of $p \in S$ in direction of the standard vector $\mathbf{e_i}$ is the piece of ray $p + \lambda\mathbf{e_i}, \lambda \geqslant 0$, which follows a crease of $\mathfrak{S}_S$. If $p \vee q$ is equal to $p + \lambda\mathbf{e_i}$, for some $\lambda \geqslant 0$, we say that it is an elbow of *type* i. The corresponding elbow geodesic is also said to be of type i. Observe that $p \vee q$ shares two coordinates with at least one (and perhaps both) of p and q. We say that a geodesic elbow is *uni-directed* if its corresponding elbow $p \vee q$ shares two of its

[3] I.e., $\forall p' = (p'_1, p'_2, p'_3) \in \mathcal{P}'$, $p'_1\mathbf{e_1} + p'_2\mathbf{e_2} + p'_3\mathbf{e_3} = 0$.

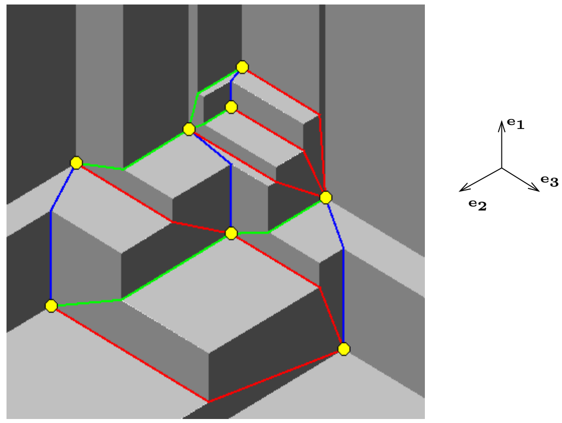

Figure 2: A coplanar orthogonal surface with its geodesic embedding.

coordinates either with p or with q (but not with both).

An orthogonal surface $\mathfrak{S}_S$ is *uni-directed* if all the geodesic elbows are uni-directed.

Definition 2 *Let S be a set of points on $\mathcal{P}$ such that the orthogonal surface $\mathfrak{S}_S$ is uni-directed. The* geodesic embedding *of S is the directed graph $\overrightarrow{\mathbf{Geo}}(S)$ defined as follows:*

- *vertices of $\overrightarrow{\mathbf{Geo}}(S)$ are the points of S.*
- *there is an arc from p to q colored i if and only if $p \vee q$ is an elbow of type i.*

2.3 TD-Delaunay triangulation

We recall here the definition of TD-Delaunay graphs introduced in [Che89].

Let T (resp. $\tilde{T}$) be the equilateral triangle of size length 1 whose barycenter is the origin and one of its vertices is on the positive (resp. negative) y-axis . A *homothet* of T is obtained by scaling T with respect to the origin, followed by a translation: $p + \lambda T = \{p + \lambda z : z \in T\}$. The *triangular distance* between two points p and q is defined as follows:

$$d_T(p,q) = \min\{\lambda : \lambda \geqslant 0 \text{ and } q \in p + \lambda T\}$$

Notice that in general $d_T(p,q) \neq d_T(q,p)$.

Let S be a set of points in the plane $\mathcal{P}$. For each $p \in S$, we define the *TD-Voronoi cell of p* as:

$$V_T(p) = \{x \in \mathcal{P} : \text{ for all } q \in S, d_T(p,x) \leqslant d_T(q,x)\} \ .$$

Fig. 3(a) shows an example of set of TD-Voronoi cells, also called TD-Voronoi diagram.

Observe that the intersection of two TD-Voronoi cells may have a positive area. For instance, consider the following set $S = \{u = (-\sqrt{3}, 1), v = (\sqrt{3}, 1)\}$ (see Fig. 3(b)). The intersection $V_T(u) \cap V_T(v)$ contains the part of the plane below the two lines (o,u) and (o,v) where $o = (0,0)$.

We say that a set of points S is *non-ambiguous* if the intersection of any two TD-Voronoi cells of S is of null area[4].

Definition 3 *Let S be a non-ambiguous set of points of $\mathcal{P}$. The TD-Delaunay graph of S, denoted by* $\mathbf{TDDel}(S)$*, is defined as follows:*

- *vertex set of* $\mathbf{TDDel}(S)$ *is S; and*
- *(p,q) is an edge of* $\mathbf{TDDel}(S)$ *if and only if $V_T(p) \cap V_T(q) \neq \varnothing$.*

[4]For ambiguous set of points S, it is possible to have a partition of the plane by the interior of Voronoi cells plus the union of all boundaries, by ordering the elements of S to break ties. See for example [BCCS08].

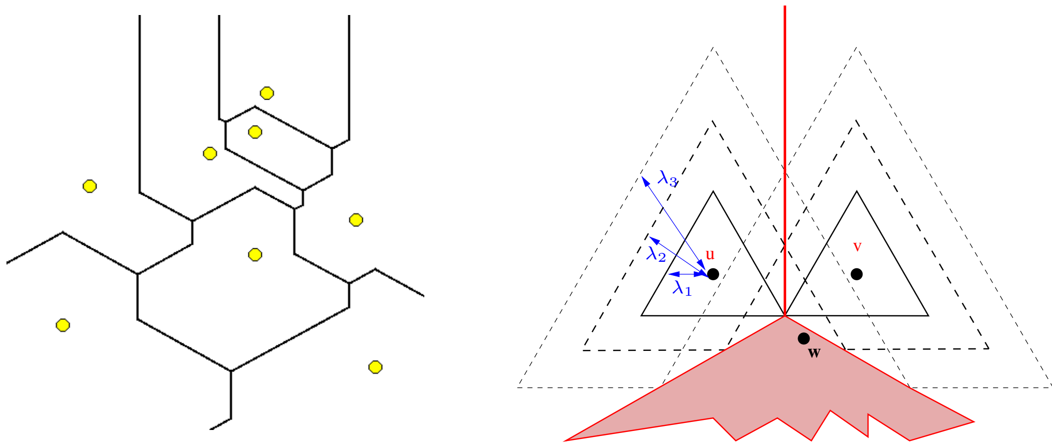

Figure 3: (a) TD-Voronoi diagram. (b) $\lambda_1 < \lambda_2 < \lambda_3$ stand for three triangular distances. Set $\{u, v\}$ is an ambiguous point set, however $\{u, v, w\}$ is non-ambiguous.

3 Unification of the concepts

3.1 Preliminaries

Given two points p and $q \in C_i^p$, we denote by $T_i(p, q)$ the set of points of $\mathcal{P}$ in $C_i^p \setminus \{p\}$ whose orthogonal projection onto $\ell_{C_i^p}$ is strictly closer to p than the orthogonal projection onto $\ell_{C_i^p}$ of q. Note that the boundary of $T_i(p, q)$ is an equilateral triangle. The interior of $T_i(p, q)$ is denoted by $T_i^\circ(p, q)$. Differently speaking, $T_i^\circ(p, q)$ is the set of points $T_i(p, q)$ deprived of the points lying on the axes of the cone C_i^p.

Lemma 1 *Let S be a set of points in the plane $\mathcal{P}$, and let p and q be two distinct points in this set. There is an arc (p, q) colored i in $\frac{1}{2}\overrightarrow{\Theta_6}(S)$ if and only if $q \in C_i^p$ and $T_i(p, q) \cap S = \varnothing$.*

Proof. This follows directly from the definition of $\frac{1}{2}\overrightarrow{\Theta_6}(S)$ (see Def. 1) and of $T_i(p, q)$. $\square$

Lemma 2 *Let S be a set of points in the plane $\mathcal{P}$, and let p and q be two distinct points in this set. $p \vee q$ is an elbow of type i if and only if $q \in C_i^p$ and $T_i^\circ(p, q) \cap S = \varnothing$.*

Proof. Let S be a set of points in the plane $\mathcal{P}$, and let p and q be two distinct points in this set. Without loss of generality, assume that p is the point with coordinates $(0, 0, 0)$ and let (q_1, q_2, q_3) be the coordinates of q. Note that the cone C_p^i can be described as the set of points $r = (r_1, r_2, r_3)$ such that $r_i \geqslant 0$ and $r_j \leqslant 0$ for all $j \neq i$. Moreover, the point p is the only point r of C_p^i such that $r_i = 0$, and the interior of the cone C_p^i is exactly the set of points r such that $r_i > 0$ and $r_j < 0$ for all $j \neq i$. These remarks are illustrated in Fig. 4. We now prove the lemma for $i = 1$. The cases $i = 2$ and $i = 3$ can be treated similarly.

By definition, $p \vee q$ is an elbow of type 1 if and only if (1) the point $p \vee q$ shares the two coordinates of index 2 and 3 with the point p, and (2) $p \vee q$ lies on the orthogonal surface $\mathfrak{S}_S$.

The point $p \vee q$ shares the two coordinates of index 2 and 3 with the point p if and only if $q_1 > 0$ and $q_2, q_3 \leqslant 0$, and thus, according to the previous remarks, if and only if $q \in C_1^p$.

Assume now that $q \in C_1^p$, that is $p \vee q = (q_1, 0, 0)$. The point $p \vee q$ lies on the orthogonal surface $\mathfrak{S}_S$ if and only if it lies on the boundary of the filter generated by S, that is, if and only if there does not exist any point $r \in S$ such that $r_1 < q_1$ and $r_2, r_3 < 0$. Since the set of points r such that $r_1 < q_1$ and $r_2, r_3 < 0$ is exactly $T_i^\circ(p, q)$, we have that $p \vee q$ lies on $\mathfrak{S}_S$ if and only if $T_i^\circ(p, q) \cap S = \varnothing$. This concludes the proof of the lemma. $\square$

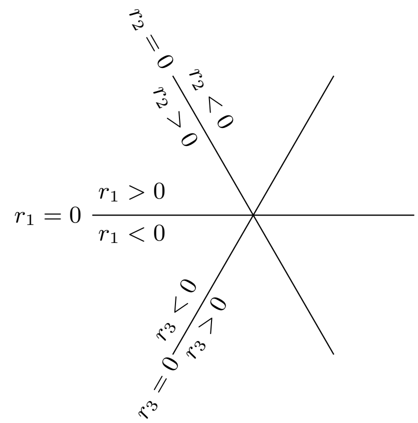

Figure 4: Partition of the plane $\mathcal{P}$ according to the sign of the coordinates of a point $r = (r_1, r_2, r_3)$ in $\mathcal{P}$.

Lemma 3 *Let S be a set of points in the plane $\mathcal{P}$, and let p and q be two distinct points in this set. The Voronoi cells $V_T(p)$ and $V_T(q)$ share at least a point if and only if there exists $i \in \{1, 2, 3\}$ such that $q \in C_i^p$ and $T_i^\circ(p, q) \cap S = \varnothing$, or $p \in C_i^q$ and $T_i^\circ(q, p) \cap S = \varnothing$.*

Proof. Let S be a set of points of $\mathcal{P}$, and let p and q be two distinct points in this set. Before proving the two parts of the equivalence, let us make the following remark. A point $r \in \mathcal{P}$ belongs to both Voronoi cells $V_T(p)$ and $V_T(q)$ if and only if there exists some $\lambda > 0$ such that both p and q lies on the boundary of $\tilde{T}' = r + \lambda\tilde{T}$ and such that the interior of $\tilde{T}'$ does not contain any point of S. Indeed the boundary of $\tilde{T}'$ consists exactly of the set of points s such that $d_T(s, r) = \lambda$ and the interior of $\tilde{T}'$ consists exactly of the set of points s such that $d_T(s, r) < \lambda$.

($\Longleftarrow$) Assume that there exists $i \in \{1, 2, 3\}$ such that $q \in C_i^p$ and $T_i^\circ(p, q) \cap S = \varnothing$, or $p \in C_i^q$ and $T_i^\circ(q, p) \cap S = \varnothing$. Without loss of generality, we have $q \in C_1^p$ and $T_i^\circ(p, q) \cap S = \varnothing$. From the previous remark, the center r of $T_i^\circ(p, q)$ belongs to both Voronoi cells $V_T(p)$ and $V_T(q)$.

($\Longrightarrow$) Assume that the Voronoi cells $V_T(p)$ and $V_T(q)$ share at least a point r. It implies that there exists some $\lambda > 0$ such that both p and q lie on the boundary of $\tilde{T}' = r + \lambda\tilde{T}$ and such that the interior of $\tilde{T}'$ does not contain any point of S.

Assume first that p and q lies on the same edge of the triangle $\tilde{T}'$. Let $\tilde{T}''$ be the positive homothet of $\tilde{T}'$ (and thus of $\tilde{T}$) having both p and q as vertices. The interior of $\tilde{T}''$ is empty because it is included in the interior of $\tilde{T}'$. Noticing that that there exists $i \in \{1, 2, 3\}$ such that $T_i^\circ(p, q)$ is exactly the interior of $\tilde{T}''$ allows to conclude the proof in this case.

Assume now that p and q lie on different edges of the triangle $\tilde{T}'$. Let s be the common point of these edges, and let $\tilde{T}''$ be the triangle obtained by a homothety of $\tilde{T}'$ of center s and of minimum ratio such that both p and q still lie on the boundary of $\tilde{T}''$. Again there exists $i \in \{1, 2, 3\}$ such that $T_i^\circ(p, q)$ or $T_i^\circ(q, p)$ is exactly the interior of $\tilde{T}''$, and this interior is empty. This concludes the proof of this lemma. $\square$

3.2 Equivalences

We first prove the links existing between the different notions of "general position" corresponding to the three objects into consideration.

Theorem 1 *Let S be a set of points in the plane $\mathcal{P}$.*

1. *S is non-degenerated if and only if $\mathfrak{S}_S$ is uni-directed.*

2. If S is non-degenerated, then S is non-ambiguous.

Proof. Given two points p and q, let $\bar{T}_i(p,q)$ be the closure of $T_i(p,q)$ deprived of the three vertices of the triangle. We will prove that the three different notions of "general position" are more or less equivalent to the fact that there do not exist two points p and q in S and $i \in \{1,2,3\}$ such that p and q are two of the vertices of $\bar{T}_i(p,q)$ and $\bar{T}_i(p,q) \cap S = \varnothing$.

Let S be a set of points in the plane $\mathcal{P}$. S is degenerated if and only if there exist two points p and q in S such that both (p,q) and (q,p) are arcs of the directed graph $\frac{1}{2}\overrightarrow{\Theta_k}(S)$. Using Lemma 1, (p,q) and (q,p) are arcs of $\frac{1}{2}\overrightarrow{\Theta_k}(S)$ of respective color i and j if and only if $\bar{T}_i(p,q) = \bar{T}_j(q,p)$ and $\bar{T}_i(p,q) \cap S = \varnothing$. To summarize, S is non-degenerated if and only if there does not exist two points p and q in S and $i \in \{1,2,3\}$ such that p and q are two of the vertices of $\bar{T}_i(p,q)$ and $\bar{T}_i(p,q) \cap S = \varnothing$.

Let S be a set of points in the plane $\mathcal{P}$. The orthogonal surface $\mathfrak{S}_S$ is not uni-directed if and only if there exist two points p and q in S such that p and q are as close as possible (using norm L_2) and such that $p \vee q$ is an elbow of type i and $q \vee p$ is an elbow of type j ($j \neq i$). Using Lemma 2, $p \vee q$ is an elbow of type i and $q \vee p$ is an elbow of type j if and only if $T_i^\circ(p,q) = T_j^\circ(q,p)$ and $T_i^\circ(p,q) \cap S = \varnothing$. Taking into account that p and q are as close as possible, we have that $\mathfrak{S}_S$ is not uni-directed if and only if there exist two points p and q in S and $i \in \{1,2,3\}$ such that p and q are two of the vertices of $\bar{T}_i(p,q)$ and $\bar{T}_i(p,q) \cap S = \varnothing$. This proves Point 1 of the Lemma.

Let S be a non-degenerated set of points in the plane $\mathcal{P}$. Assume, for the purpose of contradiction, that S is ambiguous. It means that there exist two points p and q in S such that the intersection of the two TD-Voronoi cells $V_T(p)$ and $V_T(q)$ is of non-null area. This implies that the angle between $\mathbf{e_x}$ and the line going through p and q is congruent to 0 modulo $\pi/3$. Indeed, if it is not the case, then for any $\lambda > 0$, the intersection of the two triangles $p + \lambda T$ and $q + \lambda T$ consists of at most two points, which implies that the intersection of $V_T(p)$ and $V_T(q)$ is necessarily of null area. Without loss of generality, assume that $p = (-\sqrt{3}, 1)$ and $q = (\sqrt{3}, 1)$. See Fig. 3(b) for an illustration. As shown at the beginning of the proof, the fact that S is non-degenerated implies that there exists a point $r \in S$ in $\bar{T}_3(p,q)$. By choosing p and q (with the stated properties) such that the distance between them is minimum (using L_2-norm), we can ensure that r does not lie on the segment with extremities p and q. Hence the triangular distance from r to any node in the part of the plane below the two lines (o,u) and (o,v) where $o = (0,0)$ is smaller than the triangular distance from p or q. Since this part of the plane contains the interior of the intersection $V_T(p) \cap V_T(q)$, this intersection must be of null area. This contradiction with the fact that S is ambiguous concludes the proof. $\square$

We are now ready to prove our main equivalence theorem.

Theorem 2 *Let S be a non-degenerated set of points in the plane $\mathcal{P}$. Let $\mathbf{Geo}(S)$, resp. $\frac{1}{2}\Theta_6(S)$, be the underlying undirected uncolored graph of $\overrightarrow{\mathbf{Geo}}(S)$, resp. $\frac{1}{2}\overrightarrow{\Theta_6}(S)$.*

$$\frac{1}{2}\Theta_6(S) = \mathbf{Geo}(S) = \mathbf{TDDel}(S) \ .$$

Moreover, we have

$$\overrightarrow{\mathbf{Geo}}(S) = \frac{1}{2}\overrightarrow{\Theta_6}(S) \ .$$

Proof. Let S be a non-degenerated set of points in the plane $\mathcal{P}$. From Theorem 1, all the graphs appearing in the statement of the theorem are well defined. From Lemma 2 and Lemma 3, we immediately have $\mathbf{Geo}(S) = \mathbf{TDDel}(S)$. It remains to show $\overrightarrow{\mathbf{Geo}}(S) = \frac{1}{2}\overrightarrow{\Theta_6}(S)$ to conclude the proof.

Let p and q be any two distinct points in the non-degenerated set S such that $p \vee q$ is an elbow of type i. From Lemma 2, we thus have $q \in C_i^p$. Assume, for the purpose of contradiction, that

$T_i^\circ(p,q) \cap S = \varnothing$ but $T_i(p,q) \cap S \neq \varnothing$. Let r be any point of S in $T_i(p,q) \setminus T_i^\circ(p,q)$. We have that $r \in C_i^p$ and $T_i^\circ(p,r) \cap S = \varnothing$ but we also have $p \in C_j^r$ and $T_j^\circ(r,p) \cap S = \varnothing$, for some $j \neq i$. Thus, from Lemma 2, $p \vee r$ is an elbow of type i and an elbow of type j, which contradicts the fact that S is non-degenerated, i.e., that $\mathfrak{S}_S$ is uni-directed. Thanks to this proof by contradiction, we have that two distinct points p and q in a non-degenerated set S satisfy that $p \vee q$ is an elbow of type i if and only if $q \in C_i^p$ and $T_i(p,q) \cap S = \varnothing$. From Definition 2 and Lemma 1, we have that there is an arc (p,q) colored i in $\frac{1}{2}\overrightarrow{\Theta_6}(S)$ if and only if there is an arc (p,q) colored i in $\overrightarrow{\mathbf{Geo}}(S)$. This concludes the proof of the theorem. $\square$

4 Applications

4.1 Spanner

In [Che89] it is shown that TD-Delaunay triangulations are 2-spanners. From Theorem 2 we directly get the following corollary:

Corollary 1 *Every half-Θ_6-graph (and also Θ_6-graph) is a 2-spanner. Moreover the edges of Θ_6-graph can be partitioned into two planar graphs.*

We observe that the bound of 2 is best possible stretch for half-Θ_6-graphs. Indeed, the half-Θ_6-graph of some 3-gons (the apex of a quasi-equilateral triangle) has stretch arbitrarily close to 2.

4.2 Delaunay realizability

Using face counting algorithm introduced by Schnyder [Sch89], Felsner and Zickfeld [FZ08, Theorem 10] showed that for every plane triangulation G, a point set S such that $\mathbf{Geo}(S) = G$ can be computed in linear time[5]. Using the equivalence between geodesic embeddings and TD-Delaunay triangulations (Theorem 2) we directly get the following result (see Section 1.3 for the definition of realizability):

Corollary 2 *Every plane triangulation is TD-Delaunay realizable.*

This raises the following natural question: is there another distance function Γ such that every triangulation is Γ-Delaunay realizable? The first natural distance to be considered is the L_1-norm. The next theorem shows that not all triangulations are L_1-Delaunay realizable.

Theorem 3 *The plane triangulation of K_4 is not L_1-Delaunay realizable.*

In order to prove Theorem 3 we first present a technical lemma:

Lemma 4 *Let S be a point set. Let u_1, u_2 and u_3 be three points of S. Let Δ_1 and Δ_2 be the lines going through u_1 of slope $\pi/4 \mod \pi$. These two lines split the plane around u_1 into four quater-planes. If u_2 and u_3 belong to the interior of two non-consecutive quater-planes, there is no edge (u_2, u_3) in the L_1-Delaunay graph of S.*

Proof. Let u_1, u_2 and u_3 be three points satisfying the hypothesis of the lemma. Assume without loss of generality that $u_1 = (0,0)$ and that u_2 (resp. u_3) belongs to the quater-plane containing the negative part (resp. positive part) of x-axis.

If S contains only the 3 considered vertices, one can easily check that the L_1-Voronoi cell of u_2 (resp. u_3) is included in the half-plane of strictly negative (resp. strictly positive) abscissa (see Fig. 5). Remark that adding new points to S can not extend the Voronoi cells of u_2 and u_3.

[5] Note that that this result holds also for 3-connected plane maps, but in this case the orthogonal surface is not uni-directed.

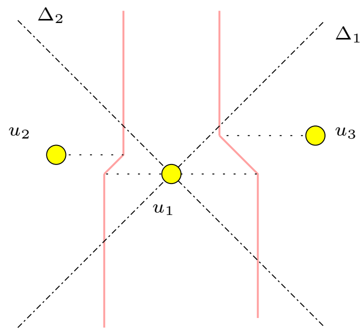

Figure 5: Impossible configuration for K_4.

Hence for any S, the Voronoi cells of u_2 and u_3 lie on disjoint part of the plane. Hence there is no edge between u_2 and u_3 in the L_1-Delaunay graph of S.

This concludes the proof of this lemma. □

Proof of Theorem 3. Let us prove by contradiction that the plane triangulation of K_4 can not be realized as a L_1-Delaunay graph.

Let $S = \{u_1, u_2, u_3, u_4\}$ such that its L_1-Delaunay graph is isomorphic to the plane triangulation of K_4. From Lemma 4 applied 3 times around u_1 considering the edges $(u_2, u_3), (u_3, u_4)$ and (u_4, u_2) we have that u_2, u_3, u_4 must all lie in at most two consecutive quater-planes. Applying this argument around each vertex shows that the four vertices are in convex position.

For convenience, assume that the vertices of the convex polygon defined by S are labeled u_1, u_2, u_3 and u_4 in the clockwise order. Hence the edges (u_1, u_3) and (u_2, u_4) cross each other in the L_1-Delanay graph. The resulting L_1-Delaunay graph corresponds to K_4 but is not a *planar* representation of K_4 because of the edge crossing. It turns out that the plane triangulation of K_4 can not be realized as L_1-Delanay graph. □

To conclude on realizability, let us mention, that there are graphs that are realizable for a certain distance function and not for another and vice versa. For instance, Theorem 3 shows that K_4 is not L_1-Delaunay realizable but it is L_2-Delaunay realizable (see the four points p_1, p_2, p_3, p_5 of Fig. 1(b) realizing K_4). On the other hand, Fig. 6 shows that there also exist graphs that are L_1-Delaunay realizable but not L_2-Delaunay realizable [Dil90, Fig. 4].

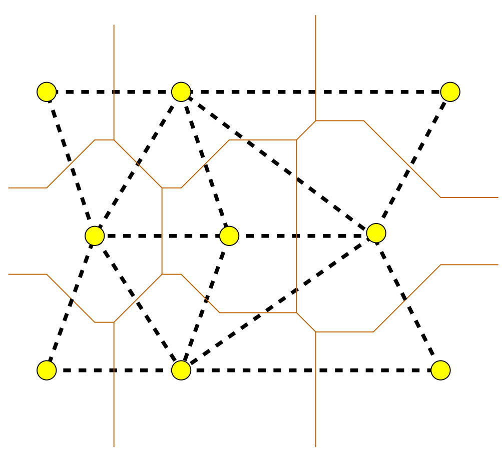

Figure 6: A triangulation that is L_1-Delaunay realizable but not L_2-Delaunay realizable.

5 Final remarks

A Voronoi diagram is sometime interpreted as a view from the top of a collection of cones whose apices lie on a plane and whose axes are oriented downward (see for example [HKL+99]). Coplanar orthogonal surfaces are the exact generalisation of this idea for TD-Voronoi diagrams: the only difference lies on the shape of the base of the cones: circular (L_2-norm), square (L_1- and L_∞-norm) or triangular (triangular distance). Hence the TD-Voronoi cell of a point p of S is exactly the orthogonal projection on $\mathcal{P}$ of the points of $\mathfrak{S}_S$ dominated by p (see figures 2 and 3).

Among various generalizations of Voronoi diagram, *Additively Weighted Voronoi Diagram* has been widely studied (see for example [DI81, KY02, BCC08]). In such a diagram, the point set is replaced by a set of weighted points $S = \{(p_1, w_1), \ldots, (p_n, w_n)\}$. The distance between an element (p_i, w_i) of S and point x of the plane $\mathcal{P}$ is $d_{AW}((p_i, w_i), x) = d(p_i, x) - w_i$. The AW-Vonoroi cell of a weighted point $(p_i, w_i) \in S$ is naturally defined as follows:

$$V_{AW}(p_i, w_i) = \{x \in \mathcal{P} : \forall (p_j, q_j) \in S, i \neq j, d_{AW}((p_i, w_i), x) \leqslant d_{AW}((p_j, w_j), x)\} \ .$$

An AW-Voronoi diagram can be interpreted as a view from the top of a collection of cones where the altitude of the apex of a cone is the weight of the corresponding element of S.

In our context we can define *Additively Weighted Triangular Distance Voronoi Diagram* (or simply *AWTD-Voronoi diagram*) using the following notion of distance: $d_{AWTD}((p_i, w_i), x) = d_T(p_i, x) - w_i$. From the previous remarks, one can see orthogonal surfaces (not necessary coplanar) as AWTD-Voronoi diagrams.

The Yao graph [Yao82] is very similar to the Θ-graph: in each cone of apex p, the selected neighbor of p is the nearest one in the cone instead of being the one with the nearest projection on ℓ_C. Half-Y_6-graphs can be defined as we did for half-Θ_6-graphs considering only 3 of the six cones. Unfortunately, half-Y_6-graphs do not have as nice structural properties. For instance, a half-Y_6-graph is not planar in general.

Algorithms that compute Θ-graphs, geodesic embeddings and TD-Delaunay triangulations have been respectively proposed in [NS07, FZ08, CD85]. It appears that the 3 proposed algorithms run in $O(n \log n)$ and are all essentially based on the “plane-sweep” algorithm.

In a forthcoming paper, the main result is used to build planar bounded-degree spanner.

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